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Ring gyroscopes: an application of adiabatic invariance

P W Forder

Physics Laboratory, University of Kent at Canterbury, UK

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Abstract. The principles of inertial rotation-sensing using ring gyroscopes are examined by applying the method of adiabatic invariance. The spinning-wheel gyroscope, the ring-laser gyro and devices containing superconducting rings are shown to be practical realisations of a theoretically modelled gyroscope.

1. Introduction

Although originally applied only to that ever popular toy (and its modern refinements used in inertial navigation systems), the term gyroscope has now come to mean any device capable of sensing inertial or 'absolute' rotations. Such devices operate in a variety of ways but all rely in some way on their intrinsic angular momentum, as in, for example, a gimbal-mounted gyroscope whose spin axis must define a direction fixed in inertial space to which the attitude of a manoeuvring vehicle may be referred. Alternatively, by demanding that the axis of the gyroscope rotate with the vehicle (as in 'strap-down' navigation systems) a measurement of rotation rate relative to inertial space may be obtained from the forces produced.

This paper will be concerned with a particularly simple form of rotation sensor (figure 1) consisting of a rigid, closed contour traversed by essentially free particles (either classical or quantum) of constant energy and momentum. We shall term such a device a *ring gyroscope*†, a familiar example of which is the laser gyro (e.g. Aronowitz 1971) whose ring-shaped optical cavity provides a closed propagation path for the circulating photons.

Since the particles within any ring gyroscope execute a *periodic* motion, it follows that the most appropriate method of analysing such a system involves the action, defined classically for each particle in terms of its momentum p and coordinate q as

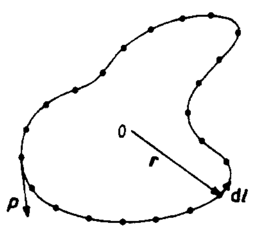

Figure 1. The model ring-gyroscope.

† It is unfortunate that this otherwise suitable term should be such a tautology (the word gyroscope is itself derived from the Greek 'gyros' meaning a ring).

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the integral over a complete cycle, $I=\oint p\,\mathrm{d}q$. By treating this quantity as an adiabatic invariant when the gyro is subjected to a slow angular acceleration, we may determine the changes in particle motion which permit an observer on the contour to deduce its inertial rotation-rate. Such an approach depends, however, on a satisfactory generalisation of the action integral to a non-inertial frame of reference, a procedure which requires certain results from the general theory of relativity. As we shall see, provided that the particles on the contour can contribute angular momentum, the action of their motion in the rotating frame involves not only their (linear) momentum, as in an inertial frame, but also a term proportional to their energy. It is this additional term which distinguishes ring gyroscopes from other, rotation-insensitive, devices.

2. Rotating frames of reference and the general theory of relativity

Since the essential feature of inertial rotation-sensing is to make measurements in a rotating (i.e. non-inertial) frame of reference, it is convenient, though not essential (Anandan 1981), to describe such processes using the general theory of relativity. Throughout this work, therefore, the diagonal metric $g_{\mu\nu}$ (μ, $\nu=0,1,2,3$) of 'flat' or inertial space–time must be replaced by the distorted metric of a uniformly rotating frame, which includes the off-diagonal elements $g_{0i}=-(\boldsymbol{\Omega}\wedge\boldsymbol{r})_i/c$ (see appendix). Consequently, neighbouring events separated by the coordinates $\mathrm{d}x^\mu=(\mathrm{d}x^0,\mathrm{d}x^i)$ are simultaneous to any observer in the rotating frame only if they occur at coordinate times that *differ* by (Landau and Lifshitz 1971)

$$\mathrm{d}x^0=-g_{0i}\,\mathrm{d}x^i/g_{00}\qquad\text{(sum over } i=1,2,3\text{).}\tag{1}$$

The effect which this has on measurements made in the rotating frame can be determined by considering three events A, B, C, say, with respective space–time coordinates (x^0, x^i), $(x^0-g_{0i}\,\mathrm{d}x^i/g_{00}, x^i+\mathrm{d}x^i)$ and $(x^0+\mathrm{d}x^0, x^i+\mathrm{d}x^i)$. According to (1) therefore, events A and B are simultaneous whilst B and C are spatially coincident. An observer at the point $(x^i+\mathrm{d}x^i)$ who wishes to use a measuring rod to determine the distance that separates events A and C must obtain this as the proper length $\mathrm{d}l$ between simultaneous events A and B where

$$-\mathrm{d}l^2=(\mathrm{d}s_{AB})^2=(g_{ik}-g_{0i}g_{0k}/g_{00})\,\mathrm{d}x^i\,\mathrm{d}x^k\qquad(i,k=1,2,3)\tag{2}$$

and $\mathrm{d}s_{AB}$ is the interval involved. Similarly, a measurement of elapsed time by the same observer using a standard clock must be obtained as the proper time $\mathrm{d}\tau$ between spatially coincident events B and C, where

$$\mathrm{d}\tau=\mathrm{d}s_{BC}/c=(g_{00}^{1/2}/c)(\mathrm{d}x^0+g_{0i}\,\mathrm{d}x^i/g_{00})\tag{3}$$

and $\mathrm{d}s_{BC}$ is the interval.

For the particular case of a frame of reference rotating with a uniform angular velocity $\boldsymbol{\Omega}$ (appendix) these expressions may be greatly simplified when $\Omega r/c$ is small and only terms to first order in Ω need be retained. Thus, equation (2) for the proper distance reduces to the flat space–time result

$$\mathrm{d}l^2\approx-g_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k=\sum_i(\mathrm{d}x^i)^2\qquad(i,k=1,2,3)$$

and we may therefore retain the concept of length as the magnitude of a vector $\mathbf{d}\boldsymbol{l}$

whose components are simply the coordinate differences

$$(\mathbf{d}\boldsymbol{l})_i = \mathrm{d}x^i \qquad (i = 1, 2, 3). \tag{4}$$

The same does not apply, however, to time intervals, since from (3), to the same order of approximation,

$$\mathrm{d}\tau \approx \mathrm{d}x^0/c + g_{0i}\,\mathrm{d}x^i/c \approx \mathrm{d}x^0/c - (1/c^2)(\boldsymbol{\Omega} \wedge \boldsymbol{r}) \cdot \mathbf{d}\boldsymbol{l} \tag{5}$$

and (4) has been used to substitute for $\mathrm{d}x^i$.

Thus, whilst the proper distance between two events can be considered purely in terms of the coordinate differences $\mathrm{d}x^i$, equation (5) shows that the proper time between events always differs from their coordinate time difference $\mathrm{d}x^0$ by an amount

$$\mathrm{d}\theta = \mathrm{d}\tau - \mathrm{d}x^0/c \approx -(1/c^2)(\boldsymbol{\Omega} \wedge \boldsymbol{r}) \cdot \mathrm{d}l \tag{6}$$

where $\mathbf{d}\boldsymbol{l}$ is their spatial separation. The result of this is that clocks at different points in the rotating frame can never be uniquely synchronised: synchronisation can take place along an *open* curve but for the closed geometry of the ring gyroscope, a discrepancy in the coordinate time always arises when the synchronisation procedure is returned to its starting point (Landau and Lifshitz 1971). For a contour of area $\boldsymbol{S} = \frac{1}{2}\oint \boldsymbol{r} \wedge \mathbf{d}\boldsymbol{l}$ (figure 1), this discrepancy takes the value (to first order in Ω)

$$\Delta T = -\oint \mathrm{d}\theta = (2/c^2)\boldsymbol{\Omega} \cdot \boldsymbol{S} \tag{7}$$

and it is from this quantity that the sensitivity of ring gyroscopes to inertial rotations derives.

3. The action for the model gyroscope

We begin by considering the model gyroscope (figure 1) at rest in an inertial (i.e. non-rotating) frame where the quasi-one-dimensional motion of the particles can be described by Hamilton's principal function $S(t, l)$ defined for each point at a distance l along the contour from some origin. If the particles have energy E and momentum p then the action for each particle is

$$I = \oint (\partial S/\partial l)\,\mathrm{d}l = \oint p\,\mathrm{d}l \tag{8}$$

and the period of the motion (transit time around contour) is $T = \mathrm{d}I/\mathrm{d}E$.

Although the particles in the model gyroscope are constrained to follow a definite path, such a motion can always be considered in terms of free propagation between perfectly reflecting surfaces arranged around the contour (as in the ring laser). A sufficiently large number of reflectors will always produce the required propagation path and the particles may therefore be treated as free throughout their motion. Consequently we may set $S(t, l) = pl - Et$ and obtain the action for each particle as simply

$$I = pL \tag{9}$$

where $L = \oint \mathrm{d}l$ is the length of the contour. The period of the motion is now

$$T = \mathrm{d}I/\mathrm{d}E = L\,\mathrm{d}p/\mathrm{d}E = L/v \tag{10}$$

where $v = \mathrm{d}E/\mathrm{d}p$ is the velocity of the particle.

Assume now that both the contour and an observer are accelerated to some angular velocity $\boldsymbol{\Omega}$ with respect to the inertial frame. Since the observer's new frame of reference is non-inertial, the modified motion of the particles along the contour must be considered in terms of a function $S'(x^\mu)$ of the general space–time coordinates. Generalising equation (8), the action for this motion is therefore

$$I' = \oint (\partial S'/\partial x^i)\,\mathrm{d}x^i = -\oint p_i\,\mathrm{d}x^i \qquad (i = 1, 2, 3) \tag{11}$$

where the components p_i are the spatial parts of the four-momentum $p_\mu = -\partial S'/\partial x^\mu$. Note, however, that since the gyro has experienced a genuine physical change as a result of its acceleration, the new function S' is not (except in very special circumstances) obtained merely as the appropriate coordinate transformation of the original function $S(t, l)$. The observer on the contour may interpret this acceleration either in classical terms, involving the appearance of non-inertial (i.e. centrifugal and Coriolis) forces or, as we choose to do here, in relativistic terms as a distortion of the space–time metric. From either point of view we may proceed in the same way as with any other change imposed on a periodic motion and provided that the acceleration is slow ($\dot{\Omega} \ll 1/T^2$)† then, by the usual arguments (e.g. Goldstein 1980), the action is an adiabatic invariant, with

$$I' = I. \tag{12}$$

In order to evaluate the integral in (11) we write the change in the value of the function S' as a particle propagates between points on the contour separated by the coordinate interval $\mathrm{d}x^\mu = (\mathrm{d}x^0, \mathrm{d}x^i)$ as

$$\mathrm{d}S' = \frac{\partial S'}{\partial x^\mu}\,\mathrm{d}x^\mu = \frac{\partial S'}{\partial x^0}\,\mathrm{d}x^0 + \frac{\partial S'}{\partial x^i}\,\mathrm{d}x^i \tag{13}$$

and note that a nearby observer rotating with the contour will interpret this change in terms of the motion of a particle with (proper) momentum $p' = \partial S'/\partial l$ and (proper) energy $E' = -\partial S'/\partial \tau$ where

$$\mathrm{d}S' = p'\,\mathrm{d}l - E'\,\mathrm{d}\tau \tag{14}$$

and $\mathrm{d}\tau, \mathrm{d}l$ are the proper time and distance involved (equations (4) and (5)). Using (5), however,

$$\frac{\partial S'}{\partial x^0} = \frac{\partial S'}{\partial \tau}\frac{\partial \tau}{\partial x^0} = \frac{1}{c}\frac{\partial S'}{\partial \tau} = -\frac{E'}{c} \tag{15}$$

and by combining (13) and (14),

$$(\partial S'/\partial x^i)\,\mathrm{d}x^i = p'\,\mathrm{d}l - E'\,\mathrm{d}\theta \tag{16}$$

where $\mathrm{d}\theta$ is the time difference of (6).

In any measurement of momentum using standard rods and clocks, it is, of course, the proper momentum of the particle p' which the observer on the contour obtains, rather than the coordinate momentum $p_i = -\partial S'/\partial x^i$. That these two quantities are different follows directly from (16). Since, however, the coordinate momentum is the same for an observer who remains in the original inertial frame, we may write (see

† In the ring-laser gyro, for example, $T \sim 10^{-9}$ s and so this condition need not be physically restricting.

appendix)

$$p_i = -\partial S'/\partial x^i = -\partial S'/\partial(\boldsymbol{r})_i = -m(\boldsymbol{V})_i$$

where, in the low-velocity limit, $\boldsymbol{V}$ is the velocity of the particle relative to the inertial frame and m is its mass. Consequently, setting $E' = mc^2$ and using (4) and (6), equation (16) becomes

$$m\boldsymbol{V} \cdot \mathbf{d}\boldsymbol{l} = p' \, \mathrm{d}l + m(\boldsymbol{\Omega} \wedge \boldsymbol{r}) \cdot \mathbf{d}\boldsymbol{l}.$$

Hence, as expected, we can interpret the proper momentum p' of the particle in the rotating frame as the simple product

$$p' = mv' = m|\boldsymbol{V} - \boldsymbol{\Omega} \wedge \boldsymbol{r}| = m \, \mathrm{d}l/\mathrm{d}\tau$$

where v' is the proper velocity in this frame.

Integrating (16) around the contour, the action of (11) becomes

$$I = \oint p' \, \mathrm{d}l - \oint E' \, \mathrm{d}\theta \tag{17}$$

and now involves not only the (proper) momentum (as in (8)) but also the (proper) energy. This is, of course, a direct result of the space–time mixing provided in a rotating frame by off-diagonal elements of the metric. To first order in Ω, the energy and momentum in the rotating frame may be treated as constants and the action integrated to obtain

$$I = p'L + E'\Delta T \tag{18}$$

where $\Delta T = -\oint \mathrm{d}\theta$ is the synchronisation discrepancy of (7). Thus, an observer on the contour who, prior to the angular acceleration, measured a proper momentum p for the circulating particles now, when rotating at a constant angular velocity, measures a proper momentum p'. The difference can be written, using (9), as

$$\Delta p = p' - p = -(E'/L)\,\Delta T \tag{19}$$

and although there will be a corresponding difference $\Delta E = E' - E$ in the proper energies of the particles before and after the acceleration, to first order in Ω this difference can be ignored in (19) and the momentum change written, using (7), as

$$\Delta p = -(E/L)\,\Delta T = -(2E/Lc^2)\boldsymbol{\Omega} \cdot \boldsymbol{S}. \tag{20}$$

To make use of (20) an observer need have no knowledge of p and E (the momentum and energy in the inertial frame) since the form of the equation allows us to write $\Delta p = \frac{1}{2}(p' - p'')$ and $E = \frac{1}{2}(E' + E'')$ where p'' and E'' are additional measurements obtained in the rotating frame with the sense of the gyro reversed (e.g. by rotating the contour through 180° about an axis normal to $\boldsymbol{S}$). As we shall see, the ring-laser form of gyroscope is particularly convenient in this respect since, by employing two identical but counter-rotating beams of photons, it can provide both pairs of measurements simultaneously. Other forms of ring gyro can be designed with two separate contours from which the observer can obtain equivalent measurements that avoid the necessity of physically reversing the device. In whatever form the gyro is used, however, the observer can, through application of (20), determine his inertial rotation rate Ω (or rather that component normal to the contour area) purely from measurements obtained in his own frame of reference. In this sense, therefore, the inertial rotation rate is 'absolute'.

4. The propagation of light: the Sagnac effect

In terms of the interval concerned, (13) can be rewritten as (Landau and Lifshitz 1971)

$$\mathrm{d}S' = -mc\,\mathrm{d}s \tag{21}$$

where $\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ and m is the mass of the particle. Hence for the case of light, where $\mathrm{d}S' = 0$, the action for the motion (equation (11)) can be expressed in the alternative form

$$I = -\oint (\partial S'/\partial x^0)\,\mathrm{d}x^0. \tag{22}$$

Using (15) and noting that (to first order in Ω) the transit time around the contour is $T' = \oint(\mathrm{d}x^0)/c$, the action is simply

$$I = E'T'. \tag{23}$$

Thus for adiabatic charges in the system, the energy must be inversely proportional to its period, a result which is familiar from the classical theory of the harmonic oscillator (e.g. Goldstein 1980).

Using (10) and (18) the period of a ring gyro can also be obtained in the form

$$T' = \mathrm{d}I/\mathrm{d}E' = L\,\mathrm{d}p'/\mathrm{d}E' + \Delta T = L/v' + \Delta T \tag{24}$$

where $v' = \mathrm{d}E'/\mathrm{d}p'$ is the proper velocity of the particle. In the case of light, therefore, although the proper velocity must be c for all observers, a photon nevertheless takes a different length of time to traverse the contour in a rotating frame, the period being increased or decreased by ΔT according to the sense of propagation. This is the effect first observed by Sagnac in 1913 and which now bears his name (e.g. see Post 1967).

5. Practical forms of the ring gyroscope

A number of practical rotation sensors based on a ring geometry have been described in the literature: in this section we derive the particular form of (20) which must apply to each.

5.1. The classical (spinning-wheel) gyroscope

Although the familiar mechanical gyroscope involves the rotation of a rigid body, it may nevertheless be considered in terms of a large number of circular contours, each traversed by many particles of some nominal mass m. The (proper) energy and momentum of each particle must however be related by

$$E^2 - p^2c^2 = p_\mu p^\mu = m^2c^4$$

so that any small change in momentum required by (20) must be accompanied by the energy change

$$\Delta E = E' - E = \frac{\mathrm{d}E}{\mathrm{d}p}\Delta p = \frac{pc^2}{E}\Delta p = -\frac{pc^2}{L}\Delta T = -\frac{2p}{L}\boldsymbol{\Omega}\cdot\boldsymbol{S}. \tag{25}$$

By writing the average angular momentum of each particle as

$$\boldsymbol{J} = \oint (\boldsymbol{r} \wedge \boldsymbol{p}) \cdot \mathbf{d}\boldsymbol{l} \bigg/ \oint \mathrm{d}l = (2p/L)\boldsymbol{S} \tag{26}$$

this last result can be converted to the familiar classical expression (Landau and Lifshitz 1960)

$$\Delta E = -\boldsymbol{\Omega} \cdot \boldsymbol{J}. \tag{27}$$

Equation (27) is independent of both the mass of the particle (and so is equally applicable to massless particles such as photons) and also the nature of the contour. Hence it may be generalised to the case of many particles and many contours, allowing the total energy E'_Σ of any gyroscope to be written in terms of its total angular momentum $\boldsymbol{J}_\Sigma$ as

$$E'_\Sigma = E_\Sigma - \boldsymbol{\Omega} \cdot \boldsymbol{J}_\Sigma = E_\Sigma - \Omega J_\Sigma \cos\theta \tag{28}$$

where θ is the angle between the vectors $\boldsymbol{\Omega}$ and $\boldsymbol{J}_\Sigma$. Thus in order to minimise its energy in the rotating frame, a spinning gyroscope will always attempt to align its own axis with the axis of rotation and will require an applied torque

$$\Gamma = (\mathrm{d}/\mathrm{d}\theta)E'_\Sigma = \Omega J_\Sigma \sin\theta = |\boldsymbol{\Omega} \wedge \boldsymbol{J}_\Sigma| \tag{29}$$

to resist this tendency. (An observer in the rotating frame will interpret this behaviour in terms of an equal but opposite torque generated by the gyroscope itself.)

5.2. The quantum (ring-laser) gyro

By using the function $S'(x^\mu)$ to define the phase of a single- or many-particle wavefunction on the contour

$$\psi'(x^\mu) = \psi_0 \exp[\mathrm{i}S'(x^\mu)/\hbar] \tag{30}$$

the action (equation (11)) is simply related to the total phase change ϕ around the ring,

$$I = \oint (\partial S'/\partial x^i)\, \mathrm{d}x^i = \hbar\phi. \tag{31}$$

Since this phase must be an integral multiple N say of 2π,

$$I = \hbar\phi = 2\pi N\hbar = Nh \tag{32}$$

and the action is quantised in the familiar way. The adiabatic invariance of I follows directly from this equation since only rapid changes imposed on the system will produce quantum jumps in the value of N.

Apart from the quantisation condition of (32), the treatment of the quantum ring-gyro closely follows the analysis in § 3. It is, however, more appropriate to work in terms of the (proper) wavenumber $k' = \hbar^{-1}\partial S'/\partial l$ and (proper) frequency $\omega' = -\hbar^{-1}\partial S'/\partial\tau$ of the quantum wave rather than its momentum and energy. Thus, in place of (18) we have the total phase

$$\phi = k'L + \omega'\Delta T \tag{33}$$

which, when written in terms of the (proper) wavelength $\lambda' = 2\pi/k'$,

$$N\lambda' = L + (\omega'/k')\Delta T, \tag{34}$$

shows that only in the non-rotating frame do N wavelengths fit exactly into the contour length. Following (19) and (20), the change in wavenumber measured by an observer when the gyro is rotating becomes

$$\Delta k = -(\omega'/L)\Delta T \approx -(\omega/L)\Delta T = -(2\omega/Lc^2)\boldsymbol{\Omega}\cdot\boldsymbol{S} \tag{35}$$

whilst from (25) and (27) the corresponding change in frequency is

$$\Delta\omega = -(2k/L)\boldsymbol{\Omega}\cdot\boldsymbol{S} = -(1/\hbar)\boldsymbol{\Omega}\cdot\boldsymbol{J} \tag{36}$$

where $\boldsymbol{J}$ is the angular momentum per particle in the quantum state (equation (26)). For a circular contour of radius $R = 2S/L$ and order N we obtain the familiar quantisation result for orbital angular momentum,

$$J = I/2\pi = \hbar kR = N\hbar \tag{37}$$

and a frequency shift,

$$\Delta\omega = -k\Omega R = -N\Omega. \tag{38}$$

Although the description of the gyro in terms of discrete particles has now been discarded, the period of the motion can nevertheless be defined using (24) as

$$T' = \frac{\mathrm{d}I}{\mathrm{d}E'} = \frac{\mathrm{d}\phi}{\mathrm{d}\omega'} = \frac{L\,\mathrm{d}k'}{\mathrm{d}\omega'} + \Delta T = \frac{L}{v'} + \Delta T \tag{39}$$

where $v' = \mathrm{d}\omega'/\mathrm{d}k'$ is the group velocity of the wave.

Macek and Davis (1966) were the first to demonstrate this form of inertial rotation-sensor using a ring laser. By employing two laser modes which differed only in their sense of propagation around a common optical cavity, they were able to detect the frequency shift of (36) as the beat frequency $2\Delta\omega$ between the two beams. Note that since light is involved, the total phase change around a ring (equation (33)) can now be written following (23) as

$$\phi = \omega' T'. \tag{40}$$

5.3. The inductive (superconducting) ring-gyro

Charged particles on the contour will behave differently from uncharged particles as a result of their interaction through the electromagnetic field. In particular, the expression for the action of each particle (equation (11)) must be modified to include the electromagnetic terms qA_μ where q is the particle charge and $A^\mu(x^\mu)$ the four-potential at any point. Thus, assuming a small (proper) velocity v' for the particles, the action (equation (17)) becomes

$$I = mv'L + q\oint \boldsymbol{A}'\cdot\mathbf{d}\boldsymbol{l} - \oint E'\,\mathrm{d}\theta \tag{41}$$

where $\boldsymbol{A}'$ is the (proper) vector potential experienced by an observer at each point on the contour (defined in terms of the electromagnetic contribution to the canonical proper momentum $\partial S'/\partial l$).

The integral $\oint \boldsymbol{A}'\cdot\mathbf{d}\boldsymbol{l}$ represents a proper flux enclosed by the contour which we equate with the product $L_0 i'$ of the proper current i' (representing the collective motion of the charges) and the self-inductance $L_0 = \mu_0\Lambda$ of the contour, as determined by Neumann's formula,

$$\Lambda = (4\pi)^{-1} \oint \oint (\mathbf{d}\boldsymbol{l} \cdot \mathbf{d}\boldsymbol{l}')/|\boldsymbol{r} - \boldsymbol{r}'|. \tag{42}$$

Note that the quantity Λ is a purely geometrical factor and may be regarded as the electromagnetic length of the contour, analogous to its mechanical length $L = \oint \mathrm{d}l$.

Using these quantities (41) becomes

$$I = mv'L + qL_0 i' - \oint E' \, \mathrm{d}\theta \tag{43}$$

which, by setting $i' = nqv'$ where n is the particle density (per unit proper length) and performing the time integral, can be written as

$$I = q(mL/nq^2 + L_0)i' + E'\Delta T. \tag{44}$$

The first term in parentheses is the so-called kinetic inductance of the contour, originating from the inertia of the particles, and can be discarded whenever the number of particles involved is large (as in a good conductor) so that

$$I = qL_0 i' + E'\Delta T. \tag{45}$$

Considering the original action integral (equation (11)), however, this is equivalent to retaining only the electromagnetic term

$$I = -q \oint A_i \, \mathrm{d}x^i = q\Phi \qquad (i = 1, 2, 3) \tag{46}$$

where Φ is also a quantity with the dimensions of flux. This flux must be distinguished from the product $L_0 i'$ (the proper flux) in (45) in the same way that the action in (18) differs from $p'L$. The relationship between the two quantities is obtained in terms of the Sagnac time discrepancy ΔT by combining (45) and (46) as

$$\Phi = L_0 i' + (E'/q)\Delta T. \tag{47}$$

The adiabatic invariance of the action must now imply the invariance of the flux Φ enclosed by the contour, a requirement that is more familiar under the name of Lenz's law. Note, however, that since the action is dominated by electromagnetic terms, any changes in the system will propagate to other parts of the contour with the velocity of light rather than with the velocity of the particle. Hence the characteristic period of the motion must be taken as Λ/c rather than L/v' (equation (24)). As a result the flux will be invariant for all changes which are slow compared with this, even if the charges themselves do not move.

Setting $\Phi = L_0 i$ where i is the current when the gyro is stationary, the change in current measured by an observer when the contour is rotated is

$$\Delta i = i' - i = -(E'/qL_0)\Delta T \tag{48}$$

where the energy E' must include not only the rest-mass energy of the particle but also its kinetic energy and any contribution qV' due to the local electrostatic potential. Unless the latter is exceptionally high, however, the energy of the charge carriers in a conductor will be dominated by their rest mass† and it is sufficient to substitute

† An electron has rest-mass energy equivalent to about 0.5 MeV.

$E' = mc^2$ and use (7) to rewrite (48) as

$$\Delta i = -(2m/qL_0)\boldsymbol{\Omega} \cdot \boldsymbol{S} \tag{49}$$

(cf DeWitt 1966).

Hildebrandt (1964) has employed a conductor in the form of a cylinder, rather than a ring, to demonstrate a rotation sensor based on this effect. Initially there is no current flowing but on rotating the cylinder about its axis a uniform (proper) magnetic field is generated, of magnitude

$$B = L_0\Delta i/S = -(2m/q)\Omega. \tag{50}$$

Note that although the effect is present in any conducting contour, it is not easily observable in practice without using a superconductor, where the current, once set up, will not decay. In this case, the current carriers constitute a macroscopic quantum state of the form of (30) and any flux Φ enclosed by the contour (equation (47)) must be an integer multiple of the flux quantum h/q (cf (32)).

5.4. The differential ring (transmission-line) gyro

By placing a second contour so that it is everywhere parallel to, but slightly displaced from, the first (figure 2), a structure is formed which, as regards the motion of charged particles, is no different from a closed, parallel-wire transmission-line. Since, however, the mutual inductance between the contours is now M_0, say (determined by the analogous form of (42)), the expression for the flux linking each contour must be modified. Thus labelling the contours as 1 and 2, equation (47) may be generalised to the pair of equations

$$\Phi_1 = L_0 i'_1 + M_0 i'_2 + (E'_1/q)\Delta T, \qquad \Phi_2 = L_0 i'_2 + M_0 i'_1 + (E'_2/q)\Delta T. \tag{51}$$

If the separate currents i'_1, i'_2 are now written in terms of a current i'_0 common to both contours and a differential or transmission-line current i' then

$$i'_1 = i'_0 - i', \qquad i'_2 = i'_0 + i', \tag{52}$$

and the flux difference $\Phi = \Phi_2 - \Phi_1$ is

$$\Phi = 2(L_0 - M_0)i' + V'\Delta T \tag{53}$$

where $V' = (E'_2 - E'_1)/q$ is the potential difference between the contours. This flux passes *between* the contours and so may be expressed in terms of the characteristic inductance per unit (proper) length of the transmission line $\mu_L = 2(L_0 - M_0)/L$,

$$\Phi = L\mu_L i' + V'\Delta T. \tag{54}$$

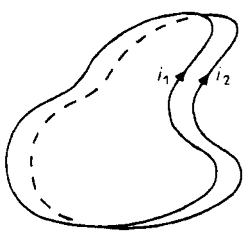

Figure 2. The differential ring-gyroscope.

Since both Φ_1, Φ_2 in (51) are adiabatically invariant then so also is Φ and if, in the inertial frame, $\Phi = L\mu_L i$ then on rotating the gyro an observer detects a change in the current of magnitude

$$\Delta i = i' - i = -V'\Delta T/\mu_L L. \tag{55}$$

Substituting for ΔT and using the relationship $\mu_L \varepsilon_L = c^{-2}$ where ε_L is the characteristic capacitance per unit (proper) length of the transmission line, (55) becomes

$$\Delta i = -(2V'\varepsilon_L/L)\boldsymbol{\Omega} \cdot \boldsymbol{S} = -(2Q'/L)\boldsymbol{\Omega} \cdot \boldsymbol{S} \tag{56}$$

where $Q' = V'\varepsilon_L$ is the stored charge per unit (proper) length of line.

By using two concentric superconducting cylinders of radius $R = 2S/L$, Brady (1981) has demonstrated a rotation sensor of this sort. The current generated when the device is rotated about its axis is (except for small corrections) directly proportional to the potential difference V' between the conductors,

$$\Delta i = -\Omega R V'\varepsilon_L = -\Omega R Q', \tag{57}$$

and is independent of the charge-to-mass ratio of the current carriers. Once again the use of superconducting materials simplifies the experiment but is in no way essential to the effect.

6. The circular ring-gyro and the invariance of action

The case of a ring gyro that is exactly circular is of particular interest since its symmetry demands that the motion of the particles along the contour remains unaffected by any rotation about its centre. Consequently, the functions S and S' must differ only as a result of the coordinate transformation between the inertial and rotating frames of reference and in fact describe the same motion, i.e. $S(t, \boldsymbol{r}) = S'(x^\mu)$ where $(t, \boldsymbol{r})$ and x^μ are the respective coordinates of the same space–time event in the two frames. Hence for the circular ring-gyro we expect the action to be not just *adiabatically* invariant but *generally* invariant under inertial rotations.

To illustrate this point, we consider the extreme case of a circular contour of radius $R = 2S/L$ and stationary particles of mass m. The period of such particles is infinite and hence the adiabatic approximation is violated for all angular accelerations. Nevertheless, the change in momentum prediction by (20),

$$\Delta p = -(E/c^2)\Omega R = -m\Omega R, \tag{58}$$

is in exact agreement with the result expected for an observer who now has a velocity ΩR with respect to the particles.

A similar argument applies to the expression for the current in a circular transmission-line (equation (57)) where we may regard the stored charges on each contour as fixed in the inertial frame. In the case of the circular ring-laser, however (equation (38)), the argument must be modified to take account of the invariant proper velocity of the photons. Here, therefore, we consider the standing wave produced by two counter-travelling beams and note that only in the inertial frame (where the frequencies are identical) does such a fringe pattern appear stationary. In the rotating frame the standing wave has an apparent velocity $-\Omega R$, so generating in a detector the interbeam beat-frequency

$$\nu_B = |2\Delta\omega/2\pi| = \Omega R/d = N\Omega/\pi \tag{59}$$

where $d = \lambda/2 = \pi/k$ is the distance between neighbouring fringes.

7. Conclusions

By considering the adiabatic invariance of the generalised action integral,

$$I = \oint (\partial S/\partial x^i)\, dx^i = -\oint p_i\, dx^i \qquad (i = 1, 2, 3),$$

together with the Sagnac time displacement around a rotating contour,

$$\Delta T = (1/c^2)\oint (\boldsymbol{\Omega} \wedge \boldsymbol{r}) \cdot \mathbf{d}\boldsymbol{l} = (2/c^2)\boldsymbol{\Omega} \cdot \boldsymbol{S},$$

the momentum change Δp for particles in a rotating ring-gyroscope of length L and area $\boldsymbol{S}$ can be found in the form

$$\Delta I = 0 = L\Delta p + E\Delta T$$

so relating the energy change $\Delta E = -\boldsymbol{\Omega} \cdot \boldsymbol{J}$ to the particle angular momentum.

When the particles constitute a macroscopic quantum state (as in the coherent photon field of the ring laser) the changes in wavenumber and frequency of the wave may be deduced from the adiabatically invariant quantum phase

$$\Delta\phi = 0 = L\Delta k + \omega\Delta T$$

whilst for particles of charge q (as in a conductor or superconductor) the change in contour current i can be obtained from the adiabatic invariance of the enclosed magnetic flux,

$$\Delta\Phi = 0 = L_0\Delta i + (E/q)\Delta T,$$

where L_0 is the inductance of the contour.

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Appendix. The metric in a rotating frame of reference

The metric $g_{\mu\nu}$ in a frame of reference rotating with constant angular velocity can be determined as follows. Consider first an inertial frame where each point is assigned spatial coordinates $\boldsymbol{r} = (x, y, z)$ and is provided with a standard clock showing the universal time t for that frame. Observers in the rotating frame can then determine their spatial coordinates x^i $(i = 1, 2, 3)$ by noting the values $(\boldsymbol{r})_i$ of the coincident point in the inertial frame when the clock at that point reads $t=0$ and their temporal coordinate x^0 by recording the reading of the inertial clock coincident with their current position (i.e. $x^0=ct$ throughout the rotating frame).

If infinitesimally separated events have coordinates $(t, \boldsymbol{r})$, $(t+\mathrm{d}t, \boldsymbol{r}+\mathrm{d}\boldsymbol{r})$ in the inertial frame and $x^\mu=(x^0, x^i)$, $x^\mu+\mathrm{d}x^\mu=(x^0+\mathrm{d}x^0,\ x^i+\mathrm{d}x^i)$ in the rotating frame then

$$\mathrm{d}x^0=c\,\mathrm{d}t \qquad \text{and} \qquad \mathrm{d}x^i=(\mathrm{d}\boldsymbol{r})_i-(\boldsymbol{\Omega}\wedge\boldsymbol{r})_i\,\mathrm{d}t$$

where $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame and an origin is chosen on the axis of rotation. Since the interval ds separating these two events must be the same in both frames (Landau and Lifshitz 1971),

$$\begin{aligned}\mathrm{d}s^2&=c^2\,\mathrm{d}t^2-|\mathrm{d}\boldsymbol{r}|^2\\&=c^2(1-u^2/c^2)\,\mathrm{d}t-2\sum_i \mathrm{d}x^i(\boldsymbol{\Omega}\wedge\boldsymbol{r})_i\,\mathrm{d}t-\sum_i(\mathrm{d}x^i)^2\\&=g_{00}(\mathrm{d}x^0)^2+2g_{0i}\,\mathrm{d}x^0\,\mathrm{d}x^i+g_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k\\&\qquad\qquad(\text{summed over } i, k=1,2,3)\\&=g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \qquad (\text{summed over } \mu, \nu=0,1,2,3)\end{aligned}$$

where $g_{\mu\nu}$ has the non-zero elements

$$g_{00}=1-u^2/c^2, \qquad g_{ii}=-1 \qquad (i=1,2,3),$$
$$g_{0i}=-(1/c)(\boldsymbol{\Omega}\wedge\boldsymbol{r})_i,$$

and $u^2=|\boldsymbol{\Omega}\wedge\boldsymbol{r}|^2$. Note that since g_{00} must be positive (Landau and Lifshitz 1971) the analysis is valid only for values of Ω and r which produce the inequality $u<c$.

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